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OPTIMAL DESIGN OF TIE-BEAMS

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Abstract—The paper concerns the determination of the optimal cross-sectional area distribution of a simply supported tie-beam that minimizes the maximum deflection, subject to: (a) a volume constraint, and (b) the longitudinal elongation of the tie-beam not exceeding a given value. The transverse load distribution considered is symmetrical about the mid-span of the tie-beam. The solution to the problem is obtained directly by two independent approaches.

1. INTRODUCTION

This paper deals with the problem of minimizing the maximum deflection of a simply supported beam of given volume (or mass) under any symmetrical transverse load distribution about its mid-span, subject to a specified limit on the longitudinal elongation should the member be used as a tie.

The problem of optimal design of multi-purpose tie-beams was first studied by Prager and Shield[1] in the late 1960s. They employed the principle of minimum potential energy in obtaining minimum-weight solutions of sandwich tie-beams for given transversal and longitudinal stiffness. Also outlined in their paper is an extension of the method to: (a) solid structures, and (b) two or more design requirements. It was only a decade later that Karihaloo and co-workers[2–4] investigated the above extensions in great detail using calculus of variations. However, their methods of solution are such that the 'response functions' of the structural member are determined only after its cross-sectional properties are obtained by an iterative procedure involving the optimality and isoperimetric conditions. Owing to the designers' preference in solving the problem through 'direct' approach, such an 'inverse' procedure suggested by Karihaloo and Parbery may not be that convenient. Consequently, Thevendran[5] presented a direct approach to the optimal design of a tie-beam under a central point load and axial tension. The approach requires the transformation of the problem into a numerical optimization one and solving it by means of a direct search technique. The aim of this paper is to complement Thevendran's work by generalizing the transverse loading distribution and considering another independent direct approach for solution. This latter approach follows closely Karihaloo's semianalytical method[2]. However, the work of Karihaloo and the present authors deviate at the point of solving the Euler equation (or optimality condition), the isoperimetric equation and the constraint equation due to the design requirement. The present method of solution involves substituting the Euler equation into the other two governing equations and directly solving them using a numerical technique for the given response function. Both direct approaches used herein give independent checks on the results obtained.

Although this paper considers only the design of tie-beams, the same procedure may be used to obtain the optimal solutions for beam-columns.

2. FORMULATION OF PROBLEM

Consider a member of length L and given volume V that has to act as a tie in some circumstances (Fig. 1(a)) and a beam in others (Fig. 1(b)). Denoting ξ as the distance along the member, measured from one end of the member, the variation of the cross-sectional area $A(\xi)$ is to be determined such that the member's maximum deflection v_{max} is a minimum under a transverse loading of intensity $2w^*(\xi)$/unit length and its longitudinal elongation does not exceed a prescribed value λ_0^* when subjected to an axial load T^*. It

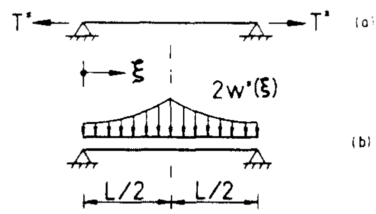

Fig. 1. Member acting as tie or beam.

is to be noted that the transverse load distribution considered is symmetrical about the mid-span of the member and hence the central deflection corresponds to the maximum deflection. In the case of nonsymmetrical applied loads, the design dependent maximum deflection may be found by the method outlined by Huang[6].

As it is convenient to deal exclusively with nondimensional quantities, the following quantities are defined:

$$x = \xi/L, \qquad \alpha(x) = A(\xi)L/V, \qquad w(x) = w^*(\xi)L^{n+3}/cEV^n,$$

$$Q = Q^*L^{n+2}/cEV^n, \qquad T = T^*L/EV, \qquad u(x) = u^*(\xi)/L, \qquad v(x) = v^*(\xi)/L$$

where ξ is the longitudinal coordinate measured from one end of the tie-beam, $A(\xi)$ the area of cross-section of the tie-beam at point ξ, L the length of the tie-beam, V the volume of the tie-beam, $2w^*(\xi)$ the intensity/unit length of the transverse distributed load carried by the tie-beam when it acts as a beam, $2Q^*$ the total transverse distributed load carried by the tie-beam when it acts as a beam, T^* the axial tensile force applied at the ends of the tie-beam when it acts as a tie, E Young's modulus of the material of the tie-beam, $u^*(\xi)$ the longitudinal displacement at point ξ when the tie-beam acts as a tie, and $v^*(\xi)$ is the transverse deflection at point ξ when the tie-beam acts as a beam. Both c and n are constants associated with the cross-sectional shapes and are defind by

$$I(\xi) = cA^n(\xi)$$

in which $I(\xi)$ is the second moment of area at ξ. Note that $n = 1$ represents a cross-section of sandwich construction; $n = 2$ or 3 represents a cross-section of solid construction ($n = 2$ corresponds to geometrically similar cross-sections while $n = 3$ corresponds to cross-sections of variable depth but of constant width).

In the formulation of the optimization problem, it is convenient to initially consider the individual objectives (beam action under a transverse load and its tie action under an axial tension) separately.

First, consider the beam action. The equilibrium equation in terms of the deflection $v^*(\xi)$ at ξ may be written in its nondimensional form as

$$(\alpha^n(x)v_{xx})_{xx} = 2w(x) \tag{1}$$

with the kinematic conditions

$$v(0) = v(1) = 0 \tag{2a}$$

$$v_{xx}(0) = v_{xx}(1) = 0 \tag{2b}$$

$$v_x(0.5) = 0 \tag{2c}$$

where the subscript x denotes differentiation with respect to x. Equations (2a) and (2b) represent the zero deflections and zero moments at the supports, respectively, while eqn

(2c) represents the zero slope at the midspan owing to the symmetry of loading and support conditions about the midspan.

Consider next the tie action. The longitudinal displacement $u^*(\xi)$ at a distance ξ from the pinned end satisfies the nondimensional differential equation

$$\alpha(x)u_x = T \tag{3}$$

with the boundary condition $u(0) = 0$.

The variation of $\alpha(x)$ is to meet the requirement

$$u(1) = 2T\int_0^{0.5} \{1/\alpha(x)\}\,\mathrm{d}x \leqslant \lambda_0$$

i.e.

$$2\int_0^{0.5} \{1/\alpha(x)\}\,\mathrm{d}x \leqslant (\lambda_0/T). \tag{4}$$

Note that owing to symmetry of loading and support conditions, we may restrict the discussion to one-half span. Since the volume of the tie-beam is given, the following isoperimetric condition is to be satisfied:

$$2\int_0^{0.5} \alpha(x)\,\mathrm{d}x = 1. \tag{5}$$

Therefore, the optimization problem under consideration consists of determining the variation of $\alpha(x)$ that: (a) satisfies the differential equations, eqns (1) and (3), (b) meets the design requirement, eqn (4), and the isoperimetric condition, eqn (5), and (c) minimizes

$$\begin{aligned} v_{\max} = v(0.5) &= \int_0^{0.5} \left(x\left\{ x\int_0^{0.5} w(\eta)\,\mathrm{d}\eta - \int_0^x (x-\eta)w(\eta)\,\mathrm{d}\eta \right\}\Big/\alpha^n(x) \right)\mathrm{d}x \\ &= \int_0^{0.5} (xf(x)/\alpha^n(x))\,\mathrm{d}x \end{aligned} \tag{6}$$

where

$$f(x) = x\int_0^{0.5} w(\eta)\,\mathrm{d}\eta - \int_0^x (x-\eta)w(\eta)\,\mathrm{d}\eta$$

gives a measure of the bending moment at point x and whilst it is dependent on the load distribution it is independent of $\alpha(x)$. Expression (6) is readily obtained by the unit-load method.

3. ANALYSIS OF POSSIBLE OPTIMAL DESIGNS

The optimization problem formulated in the preceding section involves the minimization of the objective function, expression (6), subject to the constraints given by eqns (4) and (5). The optimality condition for the optimization problem can be derived by writing an auxiliary functional

$$\Phi = \int_0^{0.5} [xf(x)/\alpha^n(x)]\,dx + \beta_1\left[1 - 2\int_0^{0.5} \alpha(x)\,dx\right] + \beta_2\left[(\lambda_0/T) - 2\int_0^{0.5} \{1/\alpha(x)\}\,dx\right] \tag{7}$$

where β_1 and β_2 are Lagrange multipliers. Euler's equation (the optimality condition) for the functional can be expressed as

$$\alpha^{n+1}(x) = c_1\alpha^{n-1}(x) + c_2 xf(x) \tag{8}$$

where c_1 and c_2 are unknown constants that include β_1 and β_2 together with some known constants. It is desirable at this juncture to examine the possibility of the existence of region in which a single-purpose optimal design may by itself meet both requirements of the design.

Consider first the optimal tie problem, i.e. the design that minimizes the longitudinal elongation of a member of given volume. The optimality condition corresponding to this problem is obtained from eqn (8) by assigning $c_2 = 0$. Thus

$$\alpha(x) = c_1^{1/2} \tag{9}$$

which on substitution into eqn (5) yields

$$\alpha(x) = 1. \tag{10}$$

Equation (3) then furnishes

$$u(1) = \lambda_{\min} = T. \tag{11}$$

Thus the optimal tie for all values of n is of constant cross-section along its length which is an obvious result. Consequently, if the prescribed value for λ_0 (the maximum permissible elongation) of the multi-purpose member in tie action satisfies the condition

$$(\lambda_0/T) \leqslant 1 \tag{12}$$

then the prismatic member itself will be of maximum longitudinal and transverse stiffness.

Next, consider the optimal beam design, i.e. the design that minimizes the maximum deflection under the transverse load. The optimality condition corresponding to this problem is obtained from eqn (8) by assigning $c_1 = 0$. Thus

$$\alpha^{n+1}(x) = c_2 xf(x)$$

or

$$\alpha(x) = \mu(xf(x))^{1/(n+1)} \tag{13}$$

where $\mu = c_2^{1/(n+1)}$, a constant which can be determined by substitution of expression (13) into eqn (5) for $n = 1, 2, 3$. From condition (4), it is clear that the optimal beam would have sufficient longitudinal stiffness to meet the requirement (4) on elongation if

$$\begin{aligned}(\lambda_0/T) &\geqslant 2\int_0^{0.5} [1/\{\mu(xf(x))^{1/(n+1)}\}]\,dx \\ &= 4\left[\int_0^{0.5} [xf(x)]^{1/(n+1)}\,dx\right]\left[\int_0^{0.5} [xf(x)]^{-1/(n+1)}\,dx\right].\end{aligned} \tag{14}$$

Hence, the region in which the optimal designs are governed by both design requirements are bounded by

$$1 < (\lambda_0/T) < \gamma(n) \tag{15}$$

where

$$\gamma(n) = 4\left[\int_0^{0.5} [xf(x)]^{1/(n+1)}\,\mathrm{d}x\right]\left[\int_0^{0.5} [xf(x)]^{-1/(n+1)}\,\mathrm{d}x\right]. \tag{16}$$

4. SPECIAL CASES

The optimization problem which has been formulated in Section 2 and analysed in Section 3 cannot be solved completely by analytical means for a general transverse distributed load when the optimal designs are governed by both design requirements, i.e. when (λ_0/T) is bounded as given by expression (15). But it may be possible to solve the problem completely by combining analytical and numerical approaches. Four simple cases are illustrated in this paper. This section deals with identifying the regions in which the optimal designs are governed by both design requirements. Subsequent sections deal with solutions of the problems presented in this section.

4.1. *Case (1). Tie-beam under a central point load and axial tension*

For this case $w(x)$ may be written as $w(x) = Q\,\delta(x - 0.5)$, where δ is the Dirac delta function. The function $f(x)$, defined in expression (6), then becomes

$$f(x) = Qx, \qquad 0 \leqslant x \leqslant 0.5. \tag{17}$$

The quantity $\gamma(n)$, given by eqn (16), becomes

$$\begin{aligned}\gamma(n) &= 4\left[\int_0^{0.5} x^{2/(n+1)}\,\mathrm{d}x\right]\left[\int_0^{0.5} x^{-2/(n+1)}\,\mathrm{d}x\right]\\ &= \begin{cases}\infty, & n = 1\\ (n+1)^2/(n+3)(n-1), & n = 2 \text{ or } 3.\end{cases}\end{aligned} \tag{18}$$

Hence, the optimal design problems governed by both design requirements, eqns (4) and (5), are to be sought only if the design parameter λ_0/T lies within the ranges

$$1 < (\lambda_0/T), \qquad n = 1 \tag{19a}$$

$$1 < (\lambda_0/T) < 1.800, \qquad n = 2 \tag{19b}$$

$$1 < (\lambda_0/T) < 1.333, \qquad n = 3. \tag{19c}$$

4.2. *Case (2). Tie-beam under a transverse uniformly distributed load and axial tension*

For this case $w(x) = Q$, a constant, and $f(x)$ is given by

$$f(x) = Qx(1 - x). \tag{20}$$

The quantity $\gamma(n)$ is furnished by

$$\begin{aligned}\gamma(n) &= 4\left(\int_0^{0.5} [x^2(1-x)]^{1/(n+1)}\,\mathrm{d}x\right)\left(\int_0^{0.5} [x^2(1-x)]^{-1/(n+1)}\,\mathrm{d}x\right)\\ &= \begin{cases}\infty, & n = 1\\ 1.667, & n = 2\\ 1.280, & n = 3\end{cases}\end{aligned} \tag{21}$$

[see Appendix A for details on evaluation of eqn (21)]. Hence, the design parameter λ_0/T

in the optimal design problem governed by both design requirements, eqns (4) and (5), is to satisfy the inequalities

$$1 < (\lambda_0/T), \qquad n = 1 \tag{22a}$$

$$1 < (\lambda_0/T) < 1.667, \qquad n = 2 \tag{22b}$$

$$1 < (\lambda_0/T) < 1.280, \qquad n = 3. \tag{22c}$$

4.3. *Case (3). Tie-beam under a transverse linearly distributed load and axial tension*

In this case, we consider a distributed load of intensity $w(x) = 8Qx$, $0 \leqslant x \leqslant 0.5$ and symmetric about the midspan. The function $f(x)$ is given by

$$f(x) = Qx\left(1 - \frac{4}{3}x^2\right), \qquad 0 \leqslant x \leqslant 0.5 \tag{23}$$

and $\gamma(n)$ is furnished by

$$\gamma(n) = 4\left(\int_0^{0.5} [x^2(1 - 4x^2/3)]^{1/(n+1)}\,\mathrm{d}x\right)\left(\int_0^{0.5} [x^2(1 - 4x^2/3)]^{-1/(n+1)}\,\mathrm{d}x\right)$$
$$= \begin{cases} \infty, & n = 1 \\ 1.732, & n = 2 \\ 1.306, & n = 3 \end{cases} \tag{24}$$

[see Appendix A for details on evaluation of eqn (24)]. Hence the design parameter λ_0/T in the optimal design problem governed by both design requirements, eqns (4) and (5), is to satisfy the inequalities

$$1 < (\lambda_0/T), \qquad n = 1 \tag{25a}$$

$$1 < (\lambda_0/T) < 1.732, \qquad n = 2 \tag{25b}$$

$$1 < (\lambda_0/T) < 1.306, \qquad n = 3. \tag{25c}$$

4.4. *Case (4). Tie-beam action under a transverse parabolically distributed load and axial tension*

In this case we consider a parabolically distributed load of intensity $w(x) = 12Qx(1 - x)$. The function $f(x)$ is given by

$$f(x) = Qx(1 - x^2)^2 \tag{26}$$

and $\gamma(n)$ is furnished by

$$\gamma(n) = 4\left(\int_0^{0.5} [x^2(1 - x^2)^2]^{1/(n+1)}\,\mathrm{d}x\right)\left(\int_0^{0.5} [x^2(1 - x^2)^2]^{-1/(n+1)}\,\mathrm{d}x\right)$$
$$= \begin{cases} \infty, & n = 1 \\ 1.704, & n = 2 \\ 1.294, & n = 3 \end{cases} \tag{27}$$

[see Appendix A for details on evaluation of eqn (27)]. Hence the design parameter λ_0/T in the optimal design problem governed by both design requirements, eqns (4) and (5), is to satisfy the inequalities

$$1 < (\lambda_0/T), \qquad n = 1 \tag{28a}$$

$$1 < (\lambda_0/T) < 1.704, \qquad n = 2 \tag{28b}$$

$$1 < (\lambda_0/T) < 1.294, \qquad n = 3. \tag{28c}$$

5. SOLUTION OF MULTI-PURPOSE OPTIMIZATION PROBLEMS

For the problems outlined in Section 4, it is not probable that closed form solution for $\alpha(x)$ exist, except when the optimal solutions are governed by only one of the two design requirements (optimal tie or optimal beam) as shown in Section 3. However, we may resort to numerical methods in determining the optimal variations of $\alpha(x)$ when both design requirements are to govern the design. Two independent approaches are presented in this paper.

5.1. *Approach I*

This semi-analytical approach involves solving the two simultaneous equations, namely, the limiting value of constraint (4) and isoperimetric condition (5) for the values of c_1 and c_2 of eqn (8) with given λ_0/T values. The expression for $\alpha(x)$ is given by the feasible solution to the quadratic, cubic and quartic equations (8) for $n = 1$, 2 and 3, respectively, namely

(for $n = 1$):
$$\alpha(x) = [c_1 + c_2 x f(x)]^{1/2} \tag{29a}$$

(for $n = 2$):
$$\alpha(x) = \begin{cases} 2(-\zeta)^{1/2}\cos[\{\cos^{-1}\omega/(-\zeta^3)\}/3] & \text{for}\,(\zeta^3 + \omega^2) \leqslant 0 \\ [\omega + (\zeta^3 + \omega^2)^{1/2}]^{1/3} + [\omega - (\zeta^3 + \omega^2)^{1/2}]^{1/3} & \text{for}\,(\zeta^3 + \omega^2) \geqslant 0 \end{cases} \tag{29b}$$

where

$$\zeta = -c_1/3 \quad \text{and} \quad \omega = c_2 x f(x)/2$$

(for $n = 3$):
$$\alpha(x) = [\{c_1 + (c_1^2 + 4c_2 x f(x))^{1/2}\}/2]^{1/2}. \tag{29c}$$

The authors have employed the Broyden linear search method[7] in solving eqns (4) and (5) for the values of c_1 and c_2. The integrals are evaluated using Simpson's rule. Once c_1 and c_2 values are known, $\alpha(x)$ is readily determined from eqns (29a)–(29c).

It should be remarked that a closed form solution for c_1 and c_2, in the sense that they can be expressed in an equation form, is obtainable for the case when the transverse load is a central point load and $n = 1$ (see Appendix B for solution).

5.2. *Approach II*

In this approach, the integrals in expressions (4)–(6) are cast into summations using Newton–Cotes formulae[8] and the problem is converted into a numerical optimization one which takes the form of:

minimizing
$$\sum_i [b_i x_i f(x_i)\alpha_i^n] \tag{30}$$

subject to
$$1 - 2\sum_i b_i \alpha_i = 0 \tag{31}$$

and
$$(\lambda_0/T) - 2\sum_i (b_i/\alpha_i) \geqslant 0 \tag{32}$$

where $\alpha_i = \alpha(x_i)$ and b_i are known constants dependent on the type of Newton–Cotes

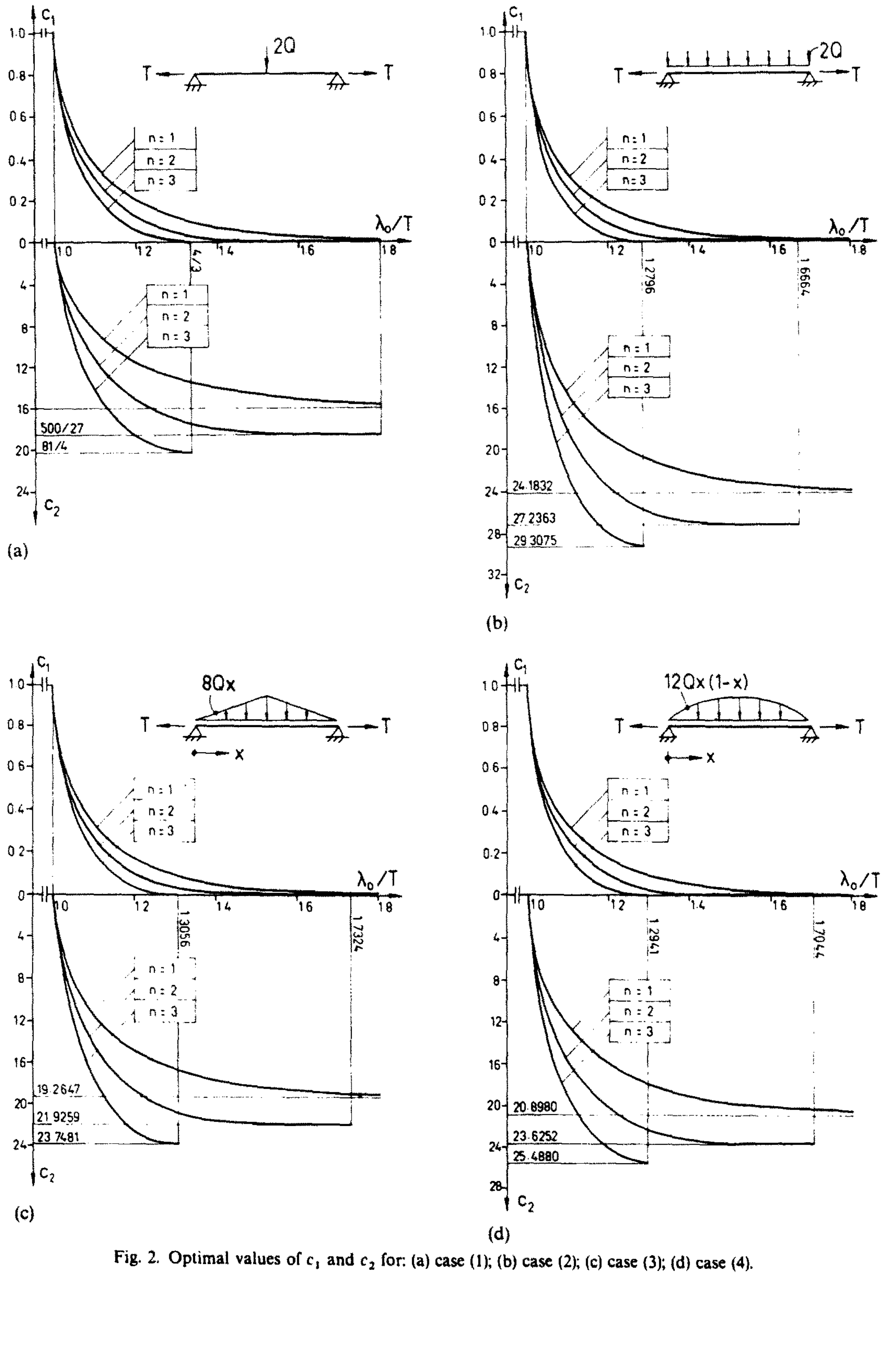

Fig. 2. Optimal values of c_1 and c_2 for: (a) case (1); (b) case (2); (c) case (3); (d) case (4).

Table 1. Variation of $\alpha(x)$. Case (1): tie-beam under a central point load and axial tension

				n = 1				
λ_0/T	1.005		1.05		1.25		1.60	
x	I	II	I	II	I	II	I	II
0	0.9172	0.9183	0.7228	0.7378	0.3943	0.4048	0.1737	0.1846
0.125	0.9338	0.9408	0.7884	0.7852	0.5912	0.5881	0.5161	0.5127
0.250	0.9819	0.9808	0.9584	0.9597	0.9653	0.9554	0.9874	0.9822
0.375	1.0573	1.0835	1.1889	1.1670	1.3791	1.3850	1.4683	1.4350
0.500	1.1546	1.1230	1.4514	1.4494	1.8057	1.7471	1.9517	1.8975
GAIN %	5.71	5.83	14.51	14.50	22.16	22.15	24.45	24.43
				n = 2				
λ_0/T	1.005		1.05		1.25		1.50	
x	I	II	I	II	I	II	I	II
0	0.9129	0.9116	0.6889	0.6929	0.2850	0.2897	0.0736	0.0760
0.125	0.9328	0.9309	0.7952	0.8011	0.6772	0.6769	0.6634	0.6649
0.250	0.9859	0.9870	0.9905	0.9901	1.0343	1.0308	1.0518	1.0203
0.375	1.0599	1.0518	1.1923	1.1931	1.3413	1.3294	1.3816	1.3645
0.500	1.1448	1.1525	1.3865	1.3740	1.6170	1.5805	1.6716	1.6080
GAIN %	10.80	10.80	25.02	24.99	34.12	34.06	35.68	35.03
				n = 3				
λ_0/T	1.005		1.05		1.15		1.25	
x	I	II	I	II	I	II	I	II
0	0.9087	0.9159	0.6522	0.6562	0.3412	0.3171	0.1308	0.1392
0.125	0.9322	0.9273	0.8122	0.8090	0.7614	0.7591	0.7514	0.7561
0.250	0.9898	0.9878	1.0135	1.0200	1.0471	1.0455	1.0586	1.0487
0.375	1.0615	1.0632	1.1879	1.1813	1.2704	1.2611	1.2949	1.2752
0.500	1.1369	1.1320	1.3414	1.3210	1.4601	1.4290	1.4942	1.4604
GAIN %	15.37	15.31	32.86	32.82	39.53	39.48	40.65	40.55

formula used. In the present study, Boole type (5-point) formula is used and the integrals reduced to summation by repeated application of the formula.

In addition to the behavioural constraints (31) and (32), side (geometric) constraints of the form

$$\alpha_1 = \alpha(0) \geqslant 0$$

and

$$\alpha_{i+1} - \alpha_i \geqslant 0, \qquad i \geqslant 1 \tag{33}$$

are introduced to obtain physically meaningful solutions.

The constrained problem is converted into an unconstrained one using the 'exterior point method' of the SUMT[9]. The resulting unconstrained problem is solved using a direct search optimization method based on Rosenbrock's method[10]. In this approach the values of $\alpha_i = \alpha(x_i)$, where $x_i = (i-1)/32$, $i = 1, 2, \ldots, 17$ are treated as the free variables of the minimization problem.

Table 2. Variation of $\alpha(x)$. Case (2): tie-beam under a transverse uniformly distributed load and axial tension

n = 1								
λ_0/T	1.005		1.05		1.25		1.60	
x	I	II	I	II	I	II	I	II
0	0.9050	0.9077	0.6902	0.7119	0.3473	0.3563	0.1347	0.1300
0.125	0.9316	0.9294	0.7913	0.7711	0.6283	0.6290	0.5814	0.5763
0.250	0.9932	1.0070	0.9950	0.9830	1.0297	1.0225	1.0558	1.0608
0.375	1.0644	1.0789	1.1997	1.1733	1.3721	1.3637	1.4403	1.4291
0.500	1.1249	1.1182	1.3586	1.3593	1.6207	1.5877	1.7154	1.6854
GAIN %	4.91	4.95	12.41	12.38	18.67	18.66	20.31	20.30
n = 2								
λ_0/T	1.005		1.05		1.25		1.50	
x	I	II	I	II	I	II	I	II
0	0.9003	0.9146	0.6527	0.6688	0.2290	0.1949	0.0611	0.0612
0.125	0.9318	0.9302	0.8091	0.7977	0.7263	0.7228	0.7219	0.7147
0.250	0.9975	0.9989	1.0229	1.0255	1.0742	1.0720	1.0866	1.0759
0.375	1.0651	1.0683	1.1930	1.1795	1.3198	1.3034	1.3449	1.3314
0.500	1.1178	1.1496	1.3104	1.3153	1.4808	1.4518	1.5112	1.4685
GAIN %	9.33	9.35	21.59	21.58	28.98	28.79	30.02	29.44
n = 3								
λ_0/T	1.005		1.05		1.15		1.25	
x	I	II	I	II	I	II	I	II
0	0.8956	0.9135	0.6110	0.6324	0.2744	0.2554	0.1145	0.1126
0.125	0.9325	0.9273	0.8335	0.8291	0.8006	0.8070	0.7965	0.7965
0.250	1.0014	0.9979	1.0399	1.0366	1.0739	1.0637	1.0813	1.0743
0.375	1.0653	1.0676	1.1830	1.1788	1.2510	1.2271	1.2643	1.2470
0.500	1.1119	1.1164	1.2759	1.2715	1.3634	1.3550	1.3802	1.3509
GAIN %	13.33	13.15	28.54	28.52	33.88	33.76	34.46	34.41

6. RESULTS AND DISCUSSION

The values obtained for parameters c_1 and c_2 of eqn (6), using Approach I outlined in Section 5.1, are given in Figs 2(a)–(d). The optimal values of $\alpha(x)$ obtained by both Approaches I and II for various cases outlined in Section 4 are tabulated in Tables 1–4 for various λ_0/T values. The results corresponding to Approaches I and II are given under columns headed by I and II, respectively, in these tables. The chosen values of λ_0/T are such that they lie in the intervals defined by inequality (15). The tables list the values of $\alpha(x)$ for $x = 0$, 0.125, 0.250, 0.375 and 0.500 only as these are intended for comparison of the results obtained by the two approaches used herein. In these tables the quantity 'GAIN %' represents the percentage gain given by

$$\mathrm{GAIN}(\%) = (1 - v_{max}/v_p)\cdot 100 \tag{34}$$

Table 3. Variation of $\alpha(x)$. Case (3): tie-beam under a transverse linearly distributed load and axial tension

n = 1								
λ_0/T	1.005		1.05		1.25		1.60	
x	I	II	I	II	I	II	I	II
0	0.9090	0.9155	0.7019	0.7231	0.3664	0.3820	0.1518	0.1407
0.125	0.9313	0.9386	0.7875	0.7729	0.6103	0.6152	0.5527	0.5458
0.250	0.9899	0.9768	0.9848	0.9869	1.0310	1.0199	1.0395	1.0273
0.375	1.0643	1.0803	1.2017	1.2178	1.3832	1.3774	1.4602	1.4541
0.500	1.1284	1.1470	1.3714	1.3516	1.6521	1.6279	1.7606	1.6976
GAIN %	5.14	5.17	13.06	13.04	19.83	19.83	21.74	21.72
n = 2								
λ_0/T	1.005		1.05		1.25		1.50	
x	I	II	I	II	I	II	I	II
0	0.9046	0.9134	0.6668	0.6627	0.2545	0.2555	0.0506	0.0694
0.125	0.9311	0.9358	0.8014	0.7925	0.7049	0.7045	0.6978	0.6952
0.250	0.9942	0.9902	1.0144	1.0049	1.0656	1.0552	1.0775	1.0537
0.375	1.0653	1.0653	1.1968	1.1899	1.3321	1.3261	1.3661	1.3567
0.500	1.1212	1.1296	1.3224	1.3340	1.5073	1.4923	1.5480	1.5249
GAIN %	9.77	9.89	22.71	22.67	30.81	30.74	32.18	31.50
n = 3								
λ_0/T	1.005		1.05		1.15		1.25	
x	I	II	I	II	I	II	I	II
0	0.9002	0.9145	0.6283	0.6357	0.3059	0.2811	0.0920	0.1245
0.125	0.9314	0.9232	0.8234	0.8274	0.7837	0.7798	0.7770	0.7840
0.250	0.9982	0.9963	1.0335	1.0289	1.0687	1.0645	1.0790	1.0736
0.375	1.0657	1.0461	1.1879	1.1760	1.2621	1.2621	1.2815	1.2631
0.500	1.1153	1.1282	1.2873	1.2826	1.3835	1.3640	1.4081	1.3880
GAIN %	13.95	13.77	30.02	29.96	35.96	35.90	36.80	36.71

where

$$v_{max} = v(0.5) = \int_0^{0.5} [xf(x)/\alpha^n(x)]\,dx$$

= midspan deflection of the designed tie-beam if it was to act as a beam

$$v_p = \int_0^{0.5} xf(x)\,dx$$

= midspan deflection of a prismatic member of the same volume if it was to act as a beam.

The quantity v_p takes the values $Q/24$, $Q/38.4$, $Q/30$ and $Q/33.02$ for the problems outlined under cases (1)–(4), respectively, in Section 4. The results for percentage gains obtained by both approaches agree very closely (within 0.5%) in all the cases considered. It is clear that a substantial gain over a prismatic member of the same volume is possible due to optimization.

Table 4. Variation of $\alpha(x)$. Case (4): tie-beam under a transverse parabolically distributed load

n = 1								
λ_0/T	1.005		1.05		1.25		1.60	
x	I	II	I	II	I	II	I	II
0	0.9053	0.9174	0.6926	0.7141	0.3542	0.3612	0.1426	0.1380
0.125	0.9306	0.9232	0.7884	0.7717	0.6208	0.6212	0.5702	0.5871
0.250	0.9939	0.9752	0.9970	1.0007	1.0336	1.0083	1.0612	1.0626
0.375	1.0666	1.0628	1.2050	1.2282	1.3813	1.3812	1.4529	1.4203
0.500	1.1181	1.1104	1.3403	1.3526	1.5935	1.5709	1.6885	1.6644
GAIN %	4.92	4.99	12.48	12.46	18.89	18.89	20.65	20.63
n = 2								
λ_0/T	1.005		1.05		1.25		1.50	
x	I	II	I	II	I	II	I	II
0	0.9009	0.9132	0.6570	0.6763	0.2411	0.2160	0.0423	0.0653
0.125	0.9307	0.9164	0.8054	0.8061	0.7180	0.7164	0.7092	0.7104
0.250	0.9982	1.0055	1.0249	1.0304	1.0785	1.0725	1.0936	1.0898
0.375	1.0670	1.0627	1.1971	1.1814	1.3265	1.3263	1.3573	1.3455
0.500	1.1119	1.1220	1.2973	1.2913	1.4644	1.4499	1.5006	1.4755
GAIN %	9.36	9.50	21.79	21.77	29.46	29.35	30.69	30.04
n = 3								
λ_0/T	1.005		1.05		1.15		1.25	
x	I	II	I	II	I	II	I	II
0	0.8964	0.9070	0.6175	0.6340	0.2900	0.2616	0.0779	0.1220
0.125	0.9314	0.9225	0.8293	0.8337	0.7941	0.7954	0.7886	0.7870
0.250	1.0021	1.0022	1.0419	1.0409	1.0775	1.0706	1.0871	1.0757
0.375	1.0669	1.0537	1.1866	1.1919	1.2571	1.2522	1.2742	1.2479
0.500	1.1067	1.1204	1.2659	1.2655	1.3534	1.3474	1.3744	1.3595
GAIN %	13.39	13.57	28.86	28.86	34.49	34.38	35.21	35.17

Of the two direct approaches described herein, Approach I provides a more accurate solution because the variation of the cross-sectional area α along the tie-beam is defined by the optimality condition [see eqns (8) and (29)]. This is confirmed in the Gain % values shown in Tables 1–4.

The methods used herein enable one to obtain directly the optimal variation of $\alpha(x)$ of a tie-beam under a transverse load distributed symmetrically about the midspan for any given value of λ_0/T for which both design requirements, eqns (4) and (5), are satisfied as equalities (i.e. when λ_0/T lies in the interval defined by inequality (15)). At the ends or outside this interval, it is possible to obtain the variation of $\alpha(x)$ analytically.

The designs considered herein are the ones with continuously varying cross-sections. Such designs may be uneconomical unless mass produced. But, when the weight consideration matters most, feasibility of such designs has to be studied. On the other hand, such designs could form the basis for comparison of other types of practicable designs.

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APPENDIX A

On evaluation of integrals appearing in expressions (21), (24) *and* (27)

The general form of the definite integrals appearing in expressions (21), (24) and (27) is

$$I(a,p,q,r) = \int_0^{0.5} x^p(1+ax^q)^r \, dx. \tag{A1}$$

Now, this integral may be evaluated after expanding the integrand $x^p(1+ax^q)^r$ using binomial expansion. Thus

$$x^p(1+ax^q)^r = x^p\left[1 + \sum_{m=1}^{\infty} \frac{r(r-1)\cdots(r-m+1)}{m!}(ax^q)^m\right] \quad \text{if } |ax^q| < 1.$$

Hence

$$\int x^p(1+ax^q)^r \, dx = x^{p+1}\left(\frac{1}{p+1} + \sum_{m=1}^{\infty} \frac{r(r-1)\cdots(r-m+1)}{m!}\frac{(ax^q)^m}{(mq+p+1)}\right) \quad \text{if } |ax^q| < 1$$

and

$$\begin{aligned} I(a,p,q,r) &= \int_0^{0.5} x^p(1+ax^q)^r \, dx \\ &= 0.5^{p+1}\left(\frac{1}{p+1} + \sum_{m=1}^{\infty} \frac{r(r-1)\cdots(r-m+1)}{m!}\frac{(0.5^q a)^m}{(mq+p+1)}\right) \\ &\quad \text{if } |ax^q| < 1 \quad \text{and} \quad p > -1. \end{aligned} \tag{A2}$$

If $p \leqslant -1$, then $I(a, p, q, r)$ would be ∞.

A computer program has been written to implement the scheme developed and hence the values of $\gamma(n)$, for $n = 2$ and 3, in expressions (21), (24) and (27), have been obtained. For $n = 1$, the values of $\gamma(n)$ become infinite.

APPENDIX B

Closed form solution for c_1 and c_2 of eqn (8) *for the tie-beam under central point load when $n = 1$*

For this case $f(x) = x$ and eqn (8) furnishes

$$\alpha(x) = (c_1 + c_2 x^2)^{1/2}.$$

Hence, eqns (4) and (5) furnish

$$2\int_0^{0.5} (c_1 + c_2x^2)^{-1/2}\,dx = \lambda_0/T \tag{B1}$$

and

$$2\int_0^{0.5} (c_1 + c_2x^2)^{1/2}\,dx = 1. \tag{B2}$$

Now

$$\int_0^{0.5} (c_1 + c_2x^2)^{-1/2}\,dx = \frac{1}{\sqrt{c_2}}\left[\ln\left(x + \sqrt{\left(x^2 + \frac{c_1}{c_2}\right)}\right)\right]_0^{0.5}$$
$$= \frac{1}{\sqrt{c_2}}\ln\left(\frac{1}{2}\sqrt{\left(\frac{c_2}{c_1}\right)} + \sqrt{\left(1 + \frac{1}{4}\frac{c_2}{c_1}\right)}\right)$$

and

$$\int_0^{0.5} (c_1 + c_2x^2)^{1/2}\,dx = \frac{\sqrt{c_2}}{2}\left[x\sqrt{\left(x^2 + \frac{c_1}{c_2}\right)} + \frac{c_1}{c_2}\ln\left(x + \sqrt{\left(x^2 + \frac{c_1}{c_2}\right)}\right)\right]_0^{0.5}$$
$$= \frac{\sqrt{c_1}}{4}\left[\sqrt{\left(1 + \frac{1}{4}\frac{c_2}{c_1}\right)} + 2\sqrt{\left(\frac{c_1}{c_2}\right)}\ln\left(\frac{1}{2}\sqrt{\left(\frac{c_2}{c_1}\right)} + \sqrt{\left(1 + \frac{1}{4}\frac{c_2}{c_1}\right)}\right)\right].$$

Hence, eqns (B1) and (B2) furnish

$$\ln\left[\frac{1}{2}\sqrt{\left(\frac{c_2}{c_1}\right)} + \sqrt{\left(1 + \frac{1}{4}\frac{c_2}{c_1}\right)}\right] = \frac{\sqrt{c_2}}{2}\frac{\lambda_0}{T} \tag{B3}$$

and

$$\sqrt{\left(1 + \frac{1}{4}\frac{c_2}{c_1}\right)} + 2\sqrt{\left(\frac{c_1}{c_2}\right)}\ln\left(\frac{1}{2}\sqrt{\left(\frac{c_2}{c_1}\right)} + \sqrt{\left(1 + \frac{1}{4}\frac{c_2}{c_1}\right)}\right) = \frac{2}{\sqrt{c_1}}. \tag{B4}$$

Equations (B3) and (B4) are valid only when both c_1 and c_2 are positive. These two equations may be solved simultaneously using Broyden linear search[7] for values of c_1 and c_2 for a given λ_0/T value.